Integer quantum Hall effect in graphene:

For this, we need to solve the Dirac eq^n in 2+1 dimensions in a magnetic field

We had found the 2+1 Dirac eq^n can be written as a matrix eq^n

$$H = -i\hbar v \begin{pmatrix} 0 & \partial x - i\partial y \\ \partial x + i\partial y & 0 \end{pmatrix} = v \begin{pmatrix} 0 & p_x - ip_y \\ p_x + ip_y & 0 \end{pmatrix}$$

Now we consider a magnetic field in the $\hat{z}$ direction $B = B_0\hat{z}$.

So we have to replace $\vec{p}$ by $\vec{p} - \frac{e\vec{A}}{c}$ $\Rightarrow$ $p_x \to p_x - \frac{eA_x}{c}$ & $p_y \to p_y - \frac{eA_y}{c}$. Let us choose $A_x = -By$, $A_y = 0$. The Landau gauge

$\Rightarrow$ $p_x \to p_x + \frac{eBy}{c}$ & $p_y \to p_y$.

In this gauge, we know the sol^{ns} are are of the form

$$\psi(x,y) = e^{ikx}\phi(y) \quad k \equiv k_x.$$

So acting $H\psi = E\psi$, we get

$$\underbrace{v_F \begin{bmatrix} 0 & \hbar\partial_y - \hbar k + \frac{Bey}{c} \\ -\hbar\partial_y - \hbar k + Bey & 0 \end{bmatrix}}_{} \phi(y) = E\,\phi(y)$$

↳ 2 comp. vector

Only diffce is that this is a matrix eq^n.

I can write this in terms of raising and lowering operators by defining

$$\xi = \frac{y}{l_B} - l_B k \qquad \text{where} \qquad l_B = \sqrt{\frac{c}{eB}}$$

$$\theta = \frac{1}{\sqrt{2}}(\partial_\xi + \xi) \quad \& \quad \theta^+ = \frac{1}{\sqrt{2}}(-\partial_\xi + \xi)$$

with $[\theta, \theta^+] = 1$ & the # ~~op~~ operator is $\theta^+\theta = N$

In terms of these operators

$H\,\phi(y) = E\,\phi(y)$ becomes

$$\omega \begin{bmatrix} 0 & \theta \\ \theta^+ & 0 \end{bmatrix} \phi(y) = E\,\phi(y)$$

with $\omega = \sqrt{2}\sqrt{\frac{eB}{c}}\,\hbar v_F$

Imp point is that $\omega \propto \sqrt{B}$, unlike in the NR case.

$$\omega \begin{bmatrix} -\lambda & \theta \\ \theta^{+} & -\lambda \end{bmatrix}$$

$$-\lambda^2 - \theta\theta^{+}$$

$$\Rightarrow \begin{bmatrix} 0 & \theta \\ \theta^{+} & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \frac{E}{\omega} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = E' \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -E' & \theta \\ \theta^{+} & -E' \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{aligned} -E'\varphi_1 + \theta\phi_2 &= 0 \\ \theta^{+}\varphi_1 - E'\varphi_2 &= 0 \end{aligned}$$

$$\Rightarrow -E'\theta^{+}\varphi_1 + \theta^{+}\theta\,\phi_2 = 0$$

$$\Rightarrow (-E'^2 + \theta^{+}\theta)\,\phi_2 = 0$$

$$\Rightarrow E'^2 = \theta^{+}\theta = N$$

$$\Rightarrow E' = \sqrt{N}$$

$$\Rightarrow E = \sqrt{N}\,\omega$$

The first point to note is that the Hamiltonian can have zero energy solutions.

$H\phi(y) = E\,\varphi(y)$ can be written as

$$(\theta\sigma^{+} + \theta^{+}\sigma^{-})\,\varphi(y) = \frac{2E}{\omega}\,\phi(y)$$

$\sigma_{+} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$\sigma_{-} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

So zero mode $\Rightarrow$

$$(\theta\sigma^+ + \theta^+\sigma^-)\,\phi(y) = 0$$

which can be satisfied; we have

$$\sigma^-\phi(y) = 0$$

$$\& \quad \theta\,\phi(y) = 0$$

(not both σ^+ & σ^-)

So $\phi(\xi) = \psi_0(\xi)\begin{pmatrix}0\\1\end{pmatrix}$

with $\theta\,\psi_0(\xi) = 0$

So $\psi_0(\xi)$ is the gnd state of the h.o.

All other states can be constructed by acting θ^+ as we did in the NR case.

We get $\phi_{N,\pm} = \begin{pmatrix}\psi_{N-1}(\xi)\\ \pm\psi_N(\xi)\end{pmatrix}$

with $E_N = \pm\omega\sqrt{N}$

with $\psi_N(\xi) = \frac{1}{\sqrt{2^{N/2}N!}}\, e^{-\xi^2/2}\, H_N(\xi)$

The main point, besides the zero energy solutions, is that one gets both +ve and −ve energy sol^{ns}. This is not surprising because we are solving the Dirac eq^{n} which has both positive and $-$ve energy sol^{ns}.

The next pt. is to find the degeneracy of these sol^{ns}. ~~Heses~~ besides the degeneracy that you have in the NR case. ~~one also~~ for graphene, one also has the degeneracy due to the the 2 valleys and also another factor of 2 because of the real spin.

So naively, one would think that we should get

$$I_{Hall} = \pm \frac{4Ne^2}{h} V$$

$+$ for electron states
$-$ for hole states

$$\Rightarrow \quad \sigma_{xy} = \pm \frac{4Ne^2}{h}$$

The zero energy sol^{n} only has $\frac{1}{2}$ the degeneracy of the other levels. So what one really ~~see~~ gets is

$$\sigma_{xy} = \pm \frac{(4N+2)e^2}{h}$$

– ie. for $N=0$, it is $\frac{\pm 2e^2}{h}$.

One way of understanding is that the $N=0$ level is equally shared by electrons and holes. Another way to see is that the zero mode is of the form $|0\rangle = \begin{pmatrix} 0 \\ \psi_0(\xi) \end{pmatrix}$ whereas all other modes have both upper and lower components — ie. they exist both on the A & B sub-lattices.

This was first exptelly seen by Geim et al, the same people who ~~fe~~ isolated graphene.